Towards constructing one-particle representations of the deformed Poincaré algebra

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Abstract

We give a method for obtaining states of massive particle representations of the two-parameter deformation of the Poincaré algebra proposed in [1], [2], [3]. We discuss four procedures to generate eigenstates of a complete set of commuting operators starting from the rest state. One result of this work is the fact that upon deforming to the quantum Poincaré algebra the rest state is split into an infinite number of states. Another result is that the energy spectrum of these states is discrete. Some curious residual degeneracy remains: there are states constructed by applying different operators to the rest state which nevertheless are indistinguishable by eigenvalues of all the observables in the algebra.

1   Introduction

Various deformations of the Poincaré algebra have been given in the literature ([4], [5], [6], [7], [8], [9], [10], [11], [12]). These deformations are speculated to play a role at Planck length scale. In this article we give procedures for constructing some states of non-zero mass representations of the two-parameter deformation of the Poincaré algebra which we proposed in [1], [2], [3].

Let us first briefly review the results of [1], [2], [3]. In [2], [3] we discussed a Lie-Poisson deformation of the Poincaré algebra. For such an algebra the Lorentz group $SL(2, C)$ does not act canonically, but instead as a Lie-Poisson group. This algebra depends on one deformation parameter λ. In [2] we showed how to quantize this algebra which, of course, introduces another parameter $\hbar$ that also can be regarded as a deformation parameter. In the limit

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$\lambda \to 0$ we recover the standard Poincaré algebra from the quantum algebra, while the limit $\hbar \to 0$ gives us the Lie-Poisson deformation introduced in [3]. In [1] we investigated properties of this two-parameter deformation of the Poincaré algebra. We found that it is covariant under $SL_q(2, C)$, determined its Casimirs and obtained the complete set of commuting operators. We also found a subalgebra which is a curious deformation of $su(2)$.

Next we summarize the results of [1] which are necessary to construct a representation. The algebra can be compactly written as follows:

$$
\begin{aligned}
\underset{12\,1\,12}{R}\underset{1}{P}\underset{12}{R^{-1}}\underset{2}{P} &= \underset{2\,21}{P}\underset{1\,21}{R^{-1}}\underset{1\,21}{P}\underset{}{R}, \\
\underset{21}{R^{-1}}\underset{1}{\Gamma}\underset{21}{R}\underset{2}{\Gamma} &= \underset{2}{\Gamma}\underset{12}{R}\underset{1}{\Gamma}\underset{12}{R^{-1}}, \\
\underset{12}{R}\underset{1}{\Gamma}\underset{12}{R^{-1}}\underset{2}{\overline{\Gamma}} &= \underset{2}{\overline{\Gamma}}\underset{12}{R}\underset{1}{\Gamma}\underset{12}{R^{-1}}, \\
\underset{21}{R^{-1}}\underset{1}{P}\underset{21}{R}\underset{2}{\Gamma} &= \underset{2}{\Gamma}\underset{21}{R^{-1}}\underset{1}{P}\underset{12}{R^{-1}},
\end{aligned}
\tag{1}
$$

We use tensor product notation, labels 1 and 2 denote different vector spaces, $\underset{1}{P} \equiv P \otimes \mathbb{I}$, $\underset{2}{P} \equiv \mathbb{I} \otimes P$, etc., $\mathbb{I}$ is 2×2 unit matrix. P, Γ and $\overline{\Gamma}$ are 2×2 matrices. P contains the four momentum operators, while Γ and $\overline{\Gamma}$ contain the angular momentum operators. Relations (1) are written out in components in the Appendix. The 2×2 matrices satisfy the following properties:

i) P is assumed to be hermitian on the states of the representation[1] we are constructing and can be expressed via components P_α of the momentum 4-vector as follows:

$$
\begin{aligned}
P &= \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = -\mathbb{I}P_0 + \sigma_k P_k \tag{2} \\
&= \begin{pmatrix} -P_0 + P_3 & P_1 - i\,P_2 \\ P_1 + i\,P_2 & -P_0 - P_3 \end{pmatrix} = P^\dagger,
\end{aligned}
$$

$$
P_\alpha = P_\alpha^\dagger, \quad \alpha = 0, 1, 2, 3. \tag{3}
$$

ii) Γ and $\overline{\Gamma}$ satisfy a deformed unimodularity condition:

$$
det_{\frac{1}{q}}(\Gamma^T) \equiv \Gamma_{11}\Gamma_{22} - q^2\Gamma_{21}\Gamma_{12} = 1,
$$

$$
det_q(\overline{\Gamma}) \equiv \overline{\Gamma}_{11}\overline{\Gamma}_{22} - \frac{1}{q^2}\overline{\Gamma}_{12}\overline{\Gamma}_{21} = 1. \tag{4}
$$

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[1]From now on we might use hermitian conjugation without mentioning the representation but, of course, without representation it is not defined, we simply do not want to write each time "on the states of our representation".

iii) Γ and $\overline{\Gamma}$ are presumed to satsify relations

$$\begin{aligned} \overline{\Gamma}^{-1} &= \Gamma^{\dagger}, \\ \Gamma^{-1} &= \overline{\Gamma}^{\dagger} \end{aligned} \tag{5}$$

and can be expressed in terms of angular momentum tensor J as follows:

$$\Gamma = e^{i\lambda J}, \quad \overline{\Gamma} = e^{i\lambda J^{\dagger}},$$

where λ is a real parameter and

$$\begin{aligned} J &= \sigma_k \left(J_{k0} - \frac{i}{2}\epsilon_{kmn} J_{mn} \right) = \tag{6} \\ &= \begin{pmatrix} -iJ_{12} + J_{30} & -iJ_{23} - iJ_{20} - J_{31} + J_{10} \\ -iJ_{23} + iJ_{20} + J_{31} + J_{10} & iJ_{12} - J_{30} \end{pmatrix}, \\ J_{\alpha\beta} &= J_{\alpha\beta}^{\dagger}, \quad \alpha, \beta = 0, 1, 2, 3. \end{aligned}$$

The $R-$ matrix is given by

$$\underset{12}{R} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{7}$$

and satisfies the quantum Yang-Baxter equation. Here $q = e^{\hbar\lambda}$ and $\hbar$ and λ can be regarded as deformation parameters. In [1] we showed that in the limit $\lambda \to 0$ we obtain the usual Poincaré algebra, while the limit $\hbar \to 0$ gives us Lie-Poisson deformation of the Poincaré algebra which we discussed in [2] and [3]. As was shown in [1], the algebra is preserved under the action of $SL_q(2, C)$:

$$\begin{aligned} P &\to P' = \overline{T}PT^{-1}, \tag{8} \\ \Gamma &\to \Gamma' = T\Gamma T^{-1}, \\ \overline{\Gamma} &\to \overline{\Gamma}' = \overline{T}\,\overline{\Gamma}\,\overline{T}^{-1}. \end{aligned}$$

where $T \in SL_q(2, C)$, $T^{\dagger} = \overline{T}^{-1}$. By definition T satisfies the commutational relations

$$\begin{aligned} \underset{12}{R}\underset{1}{T}\underset{2}{T} &= \underset{2}{T}\underset{1}{T}\underset{12}{R}, \\ \underset{12}{R}\underset{1}{T}\underset{2}{\overline{T}} &= \underset{2}{\overline{T}}\underset{1}{T}\underset{12}{R}, \tag{9} \\ \underset{12}{R}\,\underset{1}{\overline{T}}\,\underset{2}{\overline{T}} &= \underset{2}{\overline{T}}\,\underset{1}{\overline{T}}\,\underset{12}{R}, \end{aligned}$$

as well as the deformed unimodularity condition

$$det_{\frac{1}{\sqrt{q}}}(T) \equiv T_{11}T_{22} - qT_{12}T_{21} = 1 \qquad (10)$$

Our algebra appears to be distinct from systems discussed previously, e.g. in [4], [5], [6], [7], [8], [9], [10], [11], [12] and can be expressed compactly.

The following matrix was shown [1] to be an analogue of Pauli-Lubanski vector:

$$W \equiv -\mathbb{I}W_0 + \sigma_k W_k = a\left(\beta\overline{\Gamma}^{-1}P\Gamma - P\right), \qquad (11)$$

where $W^\dagger = W$, and a and β are c-numbers with the following limits: $a \underset{\hbar \to 0}{\to} \frac{1}{2\lambda}$, $\beta \underset{\hbar \to 0}{\to} 1$, $a \underset{\lambda \to 0}{\to} \frac{1}{2\lambda}$, $\beta \underset{\lambda \to 0}{\to} 1$. We shall see in Sec. 6 that $\beta = q^3$.

The algebra has two Casimir operators which are also invariant under $SL_q(2, C)$:

$$\begin{aligned} \mathcal{C}_1 &= (P, P)_q , &(12)\\ \mathcal{C}_2 &= (W, W)_q, \end{aligned}$$

where the deformed scalar product is defined as follows:

$$(A, B)_q = -\frac{1}{q^2 + 1}Tr_q\left(A\widetilde{B}\right), \qquad (13)$$

$$Tr_q(A) \equiv A_{11} + q^2 A_{22},$$

The twiddle denotes an adjugate. It is defined as follows: if B is matrix, then its adjugate $\widetilde{B}$ has the properties

$$B\widetilde{B} = \widetilde{B}B \sim \mathbb{I}$$

and in the limit $q \to 1$ it goes to $B^{-1}\det(B)$. $\widetilde{B}$ is defined uniquely up to a constant which goes to 1 in the limit $q \to 1$. For the matrices P and W we find

$$\widetilde{P} = \begin{pmatrix} P_{22} & -\frac{1}{q^2}P_{12} \\ -\frac{1}{q^2}P_{21} & \frac{1}{q^2}(P_{11} + (q^2 - 1)P_{22}) \end{pmatrix}. \qquad (14)$$

$$\widetilde{W} = \begin{pmatrix} W_{22} & -\frac{W_{12}}{q^2} \\ -\frac{W_{21}}{q^2} & \frac{W_{11}+(q^2-1)W_{22}}{q^2} \end{pmatrix} - a(q^2 - 1)\widetilde{P} \qquad (15)$$

When $\lambda \to 0$, we recover the usual form for the Pauli-Lubanski vector and Casimir operators:

$$W_\beta \underset{\lambda\to 0}{\to} -\frac{1}{2}\epsilon_\beta^{\;\mu\nu\rho} J_{\mu\nu} P_\rho, \quad \mathcal{C}_1 \underset{\lambda\to 0}{\to} P_\mu P^\mu, \quad \mathcal{C}_2 \underset{\lambda\to 0}{\to} W_\mu W^\mu.$$

From the relations (2) and (5) we are assuming that matrices P, $\Gamma + \overline{\Gamma}^{-1}$, $\overline{\Gamma} + \Gamma^{-1}$ are hermitian[2] on the states of our representations. Each hermitian matrix A gives 4 hermitian operators A_{11}, A_{22}, $A_{12} + A_{21}$, $\frac{A_{12}-A_{21}}{i}$. Therefore we have 12 hermitian operators and two constraints (4). Hence we can construct 10 independent hermitian linear combinations of the generators of our algebra. This means that by construction our representation is unitary. We will find no contradictions to this assumption later, in particular all the eigenvalues of the physically meaningful operators are real.

In section 2, 3, 4 we examine three different subalgebras of the full algebra generated by Γ, $\overline{\Gamma}$ and P. The first is a deformation of the 3-angular momentum algebra. Its representations can be easily constructed and are isomorphic to the representation of $su(2)$. In sections 3 and 4 we consider the analogues of Lorentz and Euclidean subalgebras. These subalgebras are more involved and we only give the Casimir operators for them. In section 5 we write out the complete set of commuting operators. Finally, in section 6 we find four classes of eigenstates of all the operators in the complete set of commuting operators. It is shown that in the limit $\lambda \to 0$ all the eigenvalues correspond to those of the rest state of the non-deformed Poincaré algebra. Concluding remarks are made in section 7.

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[2]where

$$\overline{\Gamma}^{-1} = \begin{pmatrix} \overline{\Gamma}_{22} & -\frac{1}{q^2}\overline{\Gamma}_{12} \\ -\frac{1}{q^2}\overline{\Gamma}_{21} & \frac{1}{q^2}\left(\overline{\Gamma}_{11} + (q^2-1)\overline{\Gamma}_{22}\right) \end{pmatrix},$$

$$\Gamma^{-1} = \begin{pmatrix} q^2\Gamma_{22} - (q^2-1)\Gamma_{11} & -q^2\Gamma_{12} \\ -q^2\Gamma_{21} & \Gamma_{11} \end{pmatrix}.$$

2 Representations of the deformed su(2) subalgebra

The central role in our construction is played by the universal enveloping algebra generated by the hermitian matrix

$$\Omega \equiv \Gamma\overline{\Gamma}^{-1} = \Gamma\Gamma^{\dagger} \tag{16}$$

which has very surprising property of commuting in the same way with all the matrices defined above:

$$Z\underset{1}{R}\Omega\underset{21}{R}=\underset{21}{R}\Omega\underset{2}{R}\underset{12}{Z}, \tag{17}$$

where $Z = P$, Γ, $\overline{\Gamma}$, W or Ω. Not all four components of Ω are independent. From the properties of Γ and $\overline{\Gamma}$ it follows that

$$det_{\frac{1}{q}}(\Omega^{T}) \equiv \Omega_{11}\Omega_{22} - q^{2}\Omega_{21}\Omega_{12} = 1. \tag{18}$$

We shall show that the representations of the algebra of Ω_{ij} are in one-to one correspondence with the representations of the $su(2)$ algebra. Furthermore, from the $\lambda \rightarrow 0$ limit

$$\frac{\Omega - \mathbb{I}}{2\lambda} \underset{\lambda\rightarrow 0}{\rightarrow} \left(\begin{array}{cc} J_{12} & J_{23} - iJ_{31} \\ J_{23} + iJ_{31} & -J_{12} \end{array} \right) \tag{19}$$

we can see that the $\Omega-$ algebra is a deformation of the $su(2)$ - algebra with Ω_{11} being analogous to the projection of 3-vector of angular momentum onto the third axis. The Casimir of the $\Omega-$ algebra is

$$Tr_{q}\left(\Omega\right) \underset{\lambda\rightarrow 0}{\rightarrow} 2 + 2\hbar\lambda + \left(\hbar^{2} - 4i\hbar J_{30} + 4\left(J_{12}^{2} + J_{23}^{2} + J_{31}^{2}\right)\right)\lambda^{2} + o\left(\lambda^{2}\right) \tag{20}$$

which corresponds to the square of 3-vector of angular momentum. [We note that although $Tr_{q}\left(\Omega\right)$ commutes with Ω_{ij}, it does not in general commute with P_{ij}, $\Gamma_{ij}$and $\overline{\Gamma}_{ij}$.] Below we shall look for states which are diagonal in Ω_{11} and $Tr_{q}\left(\Omega\right)$.

Let us rewrite the $\Omega-$ algebra (17) [with $Z = \Omega$] in components:

$$\begin{aligned} \Omega_{12}\Omega_{11} &= q^{2}\Omega_{11}\Omega_{12}, \\ \Omega_{21}\Omega_{11} &= \frac{1}{q^{2}}\Omega_{11}\Omega_{21}, \end{aligned}$$

$$\Omega_{22}\Omega_{11} = \Omega_{11}\Omega_{22},$$

$$\Omega_{22}\Omega_{12} = \Omega_{12}\Omega_{22} + \frac{q^2-1}{q^4}\Omega_{12}\Omega_{11},$$

$$\Omega_{21}\Omega_{12} = \Omega_{12}\Omega_{21} - \frac{q^2-1}{q^2}\left(\Omega_{22}-\Omega_{11}\right)\Omega_{11},$$

$$\Omega_{22}\Omega_{21} = \Omega_{21}\Omega_{22} - \frac{q^2-1}{q^2}\Omega_{21}\Omega_{11}, \tag{21}$$

$$\Omega_{11}\Omega_{22} - q^2\Omega_{21}\Omega_{12} = 1.$$

In addition the condition that Ω is hermitian means that

$$\Omega_{11} = \Omega_{11}^{\dagger}, \quad \Omega_{22} = \Omega_{22}^{\dagger}, \quad \Omega_{12} = \Omega_{21}^{\dagger}.$$

One can check that $Tr_q(\Omega)$ and Ω_{11} form a complete set of commuting operators in the $\Omega-$ algebra. From the first two commutational relations (21) it is obvious that:

$$Tr_q(\Omega)\left|k,m\right\rangle = k\left|k,m\right\rangle,$$
$$\Omega_{11}\left|k,m\right\rangle = \rho q^{2m}\left|k,m\right\rangle,$$
$$\Omega_{12}\left|k,m\right\rangle = A_{k,m}\left|k,m-1\right\rangle,$$
$$\Omega_{21}\left|k,m\right\rangle = B_{k,m}\left|k,m+1\right\rangle.$$

For simplicity we shall assume $\rho = 1$. From hermiticity of Ω :

$$A_{k,m} = B_{k,m-1}^{*}$$

From $det_{\frac{1}{q}}(\Omega^T) = 1$:

$$|A_{k,m}|^2 = \frac{q^{2m}k - q^2 - q^{4m}}{q^4},$$
$$|B_{k,m}|^2 = \frac{q^{2m}k - 1 - q^{4m+2}}{q^2}.$$

In order $|B_{k,m}|^2 \geq 0$, one must require

$$k \geq q\left(q^{2m+1} + q^{-2m-1}\right).$$

Since the right hand side monotonously increases for positive m, there exists a maximum value of m which we denote by j such that $B_{k,j} = 0$ and $k =$

$q\left(q^{2j+1}+q^{-2j-1}\right)$. Similar logic leads to the conclusion that there exists a minimum value of m and finally

$$-j \le m \le j.$$

It is therefore more convenient to use j and m to label the representation. We shall also use below the following function:

$$k_j \equiv q\left(q^{2j+1}+q^{-(2j+1)}\right). \tag{22}$$

Let us rewrite our representation in this language:

$$
\begin{aligned}
Tr_q(\Omega)\,|j,m\rangle &= k_j\,|j,m\rangle, \\
\Omega_{11}\,|j,m\rangle &= q^{2m}\,|j,m\rangle, \\
\Omega_{12}\,|j,m\rangle &= A_{j,m}\,|j,m-1\rangle, \\
\Omega_{21}\,|j,m\rangle &= B_{j,m}\,|j,m+1\rangle, \\
|B_{j,m}|^2 &= q^{2(m-1)}\left(k_j-k_m\right), \\
A_{j,m} &= B^*_{j,m-1}.
\end{aligned}
\tag{23}
$$

To see how k_j is analogous to $j(j+1)$, consider the limit:

$$a^2\left(k_j-k_0\right) \underset{\lambda\to 0}{\to} \hbar^2 j(j+1),$$

where a is the same as in (11).

Just as with the $su(2)$ representations, j can be either an integer or half-integer. Since m changes by 1 between j and $-j$ there should exist an integer n such that $j-n=-j$, that is $j=\frac{n}{2}$.

For $j=1/2$, for example, we can write out the representation of $\Omega's$ in terms of 2×2 matrices:

$$
\Omega_{11} = \begin{pmatrix} q & 0 \\ 0 & \frac{1}{q} \end{pmatrix}, \qquad
\Omega_{12} = \begin{pmatrix} 0 & 0 \\ \frac{q^2-1}{q^2} & 0 \end{pmatrix},
$$

$$
\Omega_{21} = \begin{pmatrix} 0 & \frac{q^2-1}{q^2} \\ 0 & 0 \end{pmatrix}, \qquad
\Omega_{22} = \begin{pmatrix} \frac{q^2\left(q^2-1\right)+1}{q^3} & 0 \\ 0 & q \end{pmatrix}
$$

assuming that $\left|\frac{1}{2},\frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as usual. All commutational relations and constraints are satisfied for this representation.

3   Deformed Lorentz subalgebra

The deformed Lorentz subalgebra is generated by Γ and $\overline{\Gamma}$ and contains the $\Omega-$ subalgebra. In the non-deformed case there are two Casimirs: $(J_{12}^2 + J_{23}^2 + J_{31}^2) - (J_{10}^2 + J_{20}^2 + J_{30}^2)$ and $J_{12}J_{30} + J_{23}J_{10} + J_{31}J_{20}$. In the complete set of commuting operators one can include J_{12} and J_{30} with the Casimirs. In our algebra $Tr_q\left(\Gamma + \overline{\Gamma}\right)$ and $Tr_q\left(\Gamma - \overline{\Gamma}\right)$ are Casimirs of the deformed Lorentz subalgebra. From their limits

$$Tr_q\left(\Gamma + \overline{\Gamma}\right) \underset{\lambda \to 0}{\to}$$
$$4\left(1 + \hbar\lambda + \frac{1}{2}\left(\hbar^2 - 2i\hbar J_{30} + \left(J_{12}^2 + J_{23}^2 + J_{31}^2\right) - \left(J_{10}^2 + J_{20}^2 + J_{30}^2\right)\right)\lambda^2\right)$$
$$+O\left(\lambda^3\right),$$
$$Tr_q\left(\Gamma - \overline{\Gamma}\right) \underset{\lambda \to 0}{\to} 4i\left(i\hbar J_{12} + J_{12}J_{30} + J_{23}J_{10} + J_{31}J_{20}\right)\lambda^2 + O\left(\lambda^3\right)$$

we conclude that they correspond to $(J_{12}^2 + J_{23}^2 + J_{31}^2) - (J_{10}^2 + J_{20}^2 + J_{30}^2)$ and $J_{12}J_{30} + J_{23}J_{10} + J_{31}J_{20}$, respectively. Ω_{11} is an analogue of J_{12}. However we do not know what the analogues of pure boosts are. There are several candidates that go to J_{0i} in the $\lambda \to 0$ limit and generate a closed algebra with Ω, but none of them generates a closed algebra with P and Ω, i.e. Γ and $\overline{\Gamma}$ are necessary to close the algebra.

4   Deformed Euclidean subalgebra

We denote the subalgebra generated by all $P's$ and all $\Omega's$ by $E_3^{\lambda,\hbar}$. It has the following Casimirs:

$$Tr_q(P),$$
$$(P,P)_q,$$
$$Tr_q(P\Omega) \underset{\lambda \to 0}{\to} -2P_0 + 2\lambda\left(2\overrightarrow{P}\,\overrightarrow{J} - \hbar\left(P_0 + P_3\right)\right).$$

The first Casimir is an analogue of the energy and by can be considered to be constant for an irreducible representation of the algebra. In addition to the Casimirs one can add $Tr_q(\Omega)$ and Ω_{11} to form a complete set of commuting operators for $E_3^{\lambda,\hbar}$.

5 Complete set of mutually commuting operators for the full algebra

Below we give a complete set of mutually commuting operators for our deformed Poincaré algebra:

$$
\begin{aligned}
\mathcal{C}_1 &= (P,P)_q, \qquad \mathcal{C}_2 = (W,W)_q, \\
\mathcal{K}_1 &= Tr_q(P), \\
\mathcal{K}_2 &= Tr_q(W) = a\left(\frac{\beta}{q^3}Tr_q(P\Omega) - \mathcal{K}_1\right), \\
\mathcal{K}_3 &= Tr_q(\Omega), \qquad \mathcal{K}_4 = \Omega_{11}.
\end{aligned}
\tag{24}
$$

In the $\lambda \to 0$ limit $\mathcal{K}_1$ and $\mathcal{K}_2$ go to $-2P_0$ ($\sim$ energy) and $-2W_0$ ($\sim$ projection of 3-momentum on 3-angular momentum) respectively. The limits and physical sense of the other operators in the set have been already discussed above.

6 Eigenstates

In this section we give different procedures for generating the eigenstates of our set of commuting operators.

6.1 Eigenstates of $Tr_q(\Omega)$ and Ω_{11}

Let us start by considering the eigenstates of $Tr_q(\Omega)$ and Ω_{11} constructed in section 2. If we have a set of eigenstates $\Sigma_j = \{|j,m\rangle, \quad m = -j..j, \quad Tr_q(\Omega)|j,m\rangle = k_j|j,m\rangle, \quad \Omega_{11}|j,m\rangle = q^{2m}|j,m\rangle\}$ we can obtain states with different angular momenta by applying $Z = \Gamma, \overline{\Gamma}, W, P$ to Σ_j as described below. There are four different procedures associated with the four independent operators in Z.

Procedure 1: Act with Z_{21} on the highest weight state $|j,j\rangle$:

$$
T_1 = Z_{21}|j,j\rangle.
\tag{25}
$$

From the commutational relations (17) we can compute the eigenvalues of $Tr_q(\Omega)$ and Ω_{11} :

$$
Tr_q(\Omega)T_1 = k_{j+1}T_1, \quad \Omega_{11}T_1 = q^{2(j+1)}T_1.
$$

Procedure 2: Act with $Tr_q(Z)$ on the highest weight state $|j,j\rangle$:

$$T_2 = Tr_q(Z)\,|j,j\rangle. \tag{26}$$

Then

$$Tr_q(\Omega)T_2 = k_j T_2, \quad \Omega_{11} T_2 = q^{2j} T_2$$

Procedure 3: Define (for $j \geq \frac{1}{2}$)

$$T_3 = (Z_{11} - Z_{22})\,|j,j\rangle + \frac{q^2(q^2+1)A_{j,j}}{(q^{2j}-q^{-2j})}Z_{21}\,|j,j-1\rangle. \tag{27}$$

Then

$$Tr_q(\Omega)T_3 = k_j T_3, \quad \Omega_{11} T_3 = q^{2j} T_3.$$

Procedure 4: Define (for $j \geq 1$)

$$T_4 = (Z_{11} - Z_{22})\,|j,j-1\rangle + \frac{q^3 A_{j,j-1}}{(q^{(2j-1)}-q^{-(2j-1)})}Z_{21}\,|j,j-2\rangle - \frac{q^2 B_{j,j-1}}{(q^2-1)q^{2j}}Z_{12}\,|j,j\rangle. \tag{28}$$

Then

$$Tr_q(\Omega)T_4 = k_{j-1} T_4, \quad \Omega_{11} T_4 = q^{2(j-1)} T_4.$$

Other states with fixed j but different eigenvalues of Ω_{11} can be found by acting with Ω_{12} on the highest weight T_i. We shall only consider below the eigenstates generated by the above procedures from the rest state, i.e. a state annihilated by P_{12}, P_{21} and $(P_{11} - P_{22})$. We note that acting with Γ and $\overline{\Gamma}$ can change the energy because $[\Gamma_{ij}, Tr_q(P)] \neq 0$, $[\overline{\Gamma}_{ij}, Tr_q(P)] \neq 0$. On the other hand, acting with P and W will not change the energy because $[P_{ij}, Tr_q(P)] = 0$, $[W_{ij}, Tr_q(P)] = 0$. Because we are interested in the energy spectrum we shall only consider substituting Γ and $\overline{\Gamma}$ instead of Z in the above procedures. As we shall see later, not all of the states obtained this way are eigenvalues of energy and further diagonalization is necessary. For spin 0 states to have 0 eigenvalue of $(W,W)_q$ one must also choose a particular value for β appearing in (11).

6.2 Labeling a state

Let us label an eigenstate of the operators (24) by 1) M denoting the "mass" (the eigenvalue of $(P,P)_q$ is M^2), 2) s denoting the "spin" (by this we mean that the rest state eigenvalue of $Tr_q\,(\Omega)$ is k_s), 3) the total angular momentum j (the eigenvalue of $Tr_q\,(\Omega)$), 4) "projection of total angular momentum on the third axis" m (the eigenvalue of Ω_{11} is q^{2m}). A state is thus denoted by $|M,s,j,m\rangle$. As we shall see, eigenvalues of other operators from (24) can be expressed via these 4 quantum numbers and also depend on the way in which Γ and $\overline{\Gamma}$ are applied to the rest state.

6.3 Rest state

Let us consider the highest weight rest state $|M,s,s,s\rangle$ with spin s and mass M. From the fact that P_{12}, P_{21} and $(P_{11}-P_{22})$ should annihilate this state and that the eigenvalue of $(P,P)_q$ is M^2 one can obtain:

$$
\begin{aligned}
P_{11}\,|M,s,s,s\rangle &= -M\,|M,s,s,s\rangle, \quad P_{12}\,|M,s,s,s\rangle = 0, \qquad (29)\\
P_{21}\,|M,s,s,s\rangle &= 0, \quad P_{22}\,|M,s,s,s\rangle = -M\,|M,s,s,s\rangle.
\end{aligned}
$$

From (29) and

$$
Tr_q\,(\Omega)\,|M,s,s,s\rangle = k_s\,|M,s,s,s\rangle, \quad \Omega_{11}\,|M,s,s,s\rangle = q^{2s}\,|M,s,s,s\rangle \quad (30)
$$

one can calculate eigenvalues of $Tr_q(W)$ and $(W,W)_q$. It turns out that if we want $(W,W)_q\,|M,0,0,0\rangle = 0$ and $Tr_q(W)\,|M,0,0,0\rangle = 0$ as in non-deformed case, we must fix the value of β in (11) to be

$$
\beta = q^3. \qquad (31)
$$

It is convenient for calculational purposes to rewrite $Tr_q(W)$ in the form:

$$
\begin{aligned}
\frac{Tr_q(W)}{a} &= (P_{11}-P_{22})\left(\Omega_{11} - \frac{Tr_q(\Omega)}{q^2+1}\right) + P_{12}\Omega_{21} + \\
&\quad q^2 P_{21}\Omega_{12} + Tr_q(P)\left(\frac{Tr_q(\Omega)}{q^2+1} - 1\right)
\end{aligned}
$$

and to take into account that $(W,W)_q = -a\,(q^2+1)\,(W,P)_q$ for such a choice of β (see [1] for details). Then

$$
\begin{aligned}
Tr_q(W)\,|M,s,s,s\rangle &= -Ma\,(k_s-k_0)\,|M,s,s,s\rangle, \qquad (32)\\
(W,W)_q\,|M,s,s,s\rangle &= -Ma\,(q^2+1)\,Tr_q(W)\,|M,s,s,s\rangle \\
&= -M^2 a^2 k_0\,(k_s-k_0)\,|M,s,s,s\rangle.
\end{aligned}
$$

It is now obvious that these eigenvalues vanish for zero spin. The eigenvalues have the limiting values:

$$-Ma\left(k_s - k_0\right) \underset{\lambda \to 0}{\to} -2M\hbar^2 s\left(s+1\right) \cdot \lambda \underset{\lambda \to 0}{\to} 0,$$
$$-M^2 a^2 k_0\left(k_s - k_0\right) \underset{\lambda \to 0}{\to} -M^2\hbar^2 s\left(s+1\right).$$

Note that if a state is obtained from some rest state by applying Ω_{ij} it is still a rest state due to the $\Omega - P$ commutational relations.

6.4   Procedure 1

If we apply a monomial f of l order in Γ_{21} and $\overline{\Gamma}_{21}$ to $|M, s, s, s\rangle$, we get new eigenstates of our set of commuting operators with the following eigenvalues:

$$\Omega_{11} f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle = q^{2(l+s)} f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle, \tag{33}$$
$$Tr_q\left(\Omega\right) f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle = k_{l+s} f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle, \tag{34}$$
$$Tr_q\left(P\right) f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle = -Mk_{\frac{l}{2}} f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle \tag{35}$$
$$Tr_q\left(W\right) f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle = -aM\left(k_{s+\frac{l}{2}} - k_{\frac{l}{2}}\right) \tag{36}$$
$$f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle,$$
$$\left(P_{11} - P_{22}\right) f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle = M\left(q^l - q^{-l}\right) \tag{37}$$
$$f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)|M, s, s, s\rangle.$$

Notice from (33) and (34) that l and s add like spin and orbital momentum, therefore we shall interpret l as quantum number of orbital excitations produced by $f_l\left(\Gamma_{21}, \overline{\Gamma}_{21}\right)$. Also notice from (35) that the energy

$$E_l = \frac{Mk_{\frac{l}{2}}}{k_0} \tag{38}$$

depends only on l and is discrete. In the limit $\lambda \to 0$ the energy $\to M$. As in non-deformed case $Tr_q\left(W\right)$ (an analogue of $\left(\overrightarrow{J}, \overrightarrow{P}\right)$) gives 0 when applied to spin 0 states for any value of l. For arbitrary spin s the eigenvalue of $Tr_q\left(W\right)$ goes to 0 when $\lambda \to 0$. The $\left(P_{11} - P_{22}\right)$ operator (which corresponds to the projection of the momentum on the third axis) is not, of course, in the set of commuting operators, only highest weight module is its eigenstate. In non-deformed case it has 0 eigenvalue, here it does not. Note also that

the eigenvalues are the same for any internal structure of the monomial. There seems to be no way to distinguish, for instance, $\Gamma_{21} |M, s, s, s\rangle$ from $\overline{\Gamma}_{21} |M, s, s, s\rangle$ by eigenvalues of the observables although Γ and $\overline{\Gamma}$ commute differently with other elements of the algebra and themselves.

Since

$$[\Omega_{ij}, Tr_q(Z)] = 0,$$

states with different eigenvalues of Ω_{11}, obtained by applying the lowering operator Ω_{12} to highest weight states, have the same eigenvalues of $Tr_q(\Omega)$, $Tr_q(P)$ and $Tr_q(W)$. Of course, they also have the same eigenvalues of Casimirs $(P, P)_q$ and $(W, W)_q$. The eigenvalues of Casimirs can not be changed by applying any operator to the rest state and characterize the representation of the whole algebra.

6.5 Procedures 2 and 3

This case is more complicated than the previous one because in general states diagonal in $Tr_q(\Omega)$ and Ω_{11} would not be automatically diagonal in other operators in the complete set and further diagonalization is necessary. The way diagonalization is performed depends on spin. Let us consider several cases.

6.5.1 Spin 0

In this case procedure 3 is not applicable. We can only consider procedure 2.

It can be shown that $(Tr_q(\Gamma))^n |M, 0, 0, 0\rangle$ for n greater than 1 is not an eigenstate of $Tr_q(P)$. Instead the following polynomials of $Tr_q(P)$ applied to $|M, 0, 0, 0\rangle$ are eigenstates of energy:

$$
\begin{aligned}
\pi_0 &= |M, 0, 0, 0\rangle, \\
Tr_q(P)\pi_0 &= -Mk_0\pi_0, \\
\pi_1 &= Tr_q(\Gamma) |M, 0, 0, 0\rangle, \\
Tr_q(P)\pi_1 &= -Mk_{\frac{1}{2}}\pi_1, \\
\pi_2 &= \left((Tr_q(\Gamma))^2 - q^2\right) |M, 0, 0, 0\rangle, \\
Tr_q(P)\pi_2 &= -Mk_1\pi_2, \\
\pi_3 &= \left((Tr_q(\Gamma))^3 - 2q^2 Tr_q(\Gamma)\right) |M, 0, 0, 0\rangle,
\end{aligned}
$$

$$Tr_q(P)\pi_3 = -Mk_{\frac{3}{2}}\pi_3,$$
$$\pi_4 = \left((Tr_q(\Gamma))^4 - 3q^2(Tr_q(\Gamma))^2 + q^4\right)|M,0,0,0\rangle,$$
$$Tr_q(P)\pi_4 = -Mk_2\pi_4,$$
$$\pi_5 = \left((Tr_q(\Gamma))^5 - 4q^2(Tr_q(\Gamma))^3 + 3q^4 Tr_q(\Gamma)\right)|M,0,0,0\rangle,$$
$$Tr_q(P)\pi_5 = -Mk_{\frac{5}{2}}\pi_5,$$
$$etc.$$

All these results can be compactly described by one recursive relation:

$$\pi_n - Tr_q(\Gamma)\pi_{n-1} = -q^2\pi_{n-2}, \quad \pi_0 = |M,0,0,0\rangle, \quad \pi_1 = Tr_q(\Gamma)|M,0,0,0\rangle,$$
$$Tr_q(P)\pi_n = -Mk_{\frac{n}{2}}\pi_n, \quad Tr_q(W)\pi_n = 0, \quad (W,W)_q\pi_n = 0, \qquad (39)$$
$$Tr_q(\Omega)\pi_n = k_0\pi_n, \quad \Omega_{11}\pi_n = \pi_n.$$

π_n is not an eigenstate of $(P_{11} - P_{22})$.

If one substitutes $\overline{\Gamma}$ instead of Γ everywhere in (39), the eigenvalues are unchanged. However, mixing $\overline{\Gamma}$ and Γ is much more difficult than in the case of procedure 1 since π_n is not a monomial, besides one must take into account that some quadratic combinations of $\overline{\Gamma}$ and Γ should be converted into 1 or Ω. We do not discuss this problem here.

6.5.2   Spin $1/2$

In this case procedures 2 and 3 have to be mixed and it gets even more complicated. We will consider only the first order in Γ and $\overline{\Gamma}$ case. The following are eigenstates of all the operators in the complete set of commuting operators:

$$S_1 = Tr_q(\Gamma)\left|M,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle +$$
$$q^3\left(\Gamma_{21}\left|M,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{q}\Gamma_{22}\left|M,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right),$$
$$S_2 = Tr_q(\Gamma)\left|M,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle -$$
$$q^5\left(\Gamma_{21}\left|M,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{q}\Gamma_{22}\left|M,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\right),$$
$$S_3 = Tr_q(\overline{\Gamma})\left|M,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle + \qquad (40)$$

$$q \left(\overline{\Gamma}_{21} \left| M, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle - \frac{1}{q} \overline{\Gamma}_{22} \left| M, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \right),$$

$$S_4 = Tr_q \left(\overline{\Gamma} \right) \left| M, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle -$$

$$\frac{1}{q} \left(\overline{\Gamma}_{21} \left| M, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle - \frac{1}{q} \overline{\Gamma}_{22} \left| M, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \right)$$

with eigenvalues

$$\Omega_{11} S_i = q S_i, \quad Tr_q \left(\Omega \right) S_i = k_{\frac{1}{2}} S_i, \quad Tr_q \left(P \right) S_i = -m k_{\frac{1}{2}} S_i;$$

$$Tr_q \left(W \right) S_{1,3} = a M \left(k_{\frac{1}{2}} - k_0 \right) S_{1,3}, \tag{41}$$

$$Tr_q \left(W \right) S_{2,4} = -a M \left(k_1 - k_{\frac{1}{2}} \right) S_{1,4},$$

$$\left(P_{11} - P_{22} \right) S_{1,3} = M \left(q - \frac{1}{q} \right) S_{1,3},$$

$$S_{2,4} \quad is \quad not \quad eigenstate \quad of \quad \left(P_{11} - P_{22} \right).$$

We do not know the general formula for higher orders here. Notice again that one can not distinguish here states generated by Γ and $\overline{\Gamma}$: $S_{1,3}$ have the same eigenvalues for any observables, so do $S_{2,4}$; but S_3 (S_4) is not obtained by just substituting $\overline{\Gamma}$ instead of Γ into S_1 (S_2) : the mixing coefficients are different.

6.6 Procedure 4

We only consider here how to use procedure 4 to shift from the rest state with spin 1 to the excited state with angular momentum 0.

$$S_5 = -\left(\Gamma_{11} - \Gamma_{22} \right) \left| M, 1, 1, 0 \right\rangle +$$

$$\frac{\sqrt{q^2 + 1}}{q^2} \left(\Gamma_{12} \left| M, 1, 1, 1 \right\rangle - q^3 \Gamma_{21} \left| M, 1, 1, -1 \right\rangle \right),$$

$$\Omega_{11} S_5 = S_5, \quad Tr_q \left(\Omega \right) S_5 = k_0 S_5, \tag{42}$$

$$Tr_q \left(P \right) S_5 = -M k_{\frac{1}{2}} S_5, \quad Tr_q \left(W \right) S_5 = 0.$$

Again, we can substitute $\overline{\Gamma}$ instead of Γ above.

7 Conclusion

Eigenvalues for all the states found above in the limit $\lambda \to 0$ go to those of the rest state of the non-deformed Poincaré algebra. Therefore the deformation splits the rest state into an infinite number of states with a discrete energy spectrum which is bounded from below by M but is unbounded from above. Within this interpretation, the observed curious strong degeneracy of states (when states constructed by applying algebraically independent operators to the rest state are nevertheless indistinguishable by eigenvalues of all the observables in the algebra) is not that surprising since in the limit they all have the same eigenvalues.

States generated by procedures 1 and 4 have different energies and angular momenta and hence can be thought of as "quantum rotations", while states generated by procedures 2 and 3 have different energies but same angular momentum and therefore can be interpreted as "quantum oscillations" which a particle acquires upon the deformation.

It is remarkable that both "rotations" and "oscillations" have exactly the same expression (38) for the energy spectrum (though different quantum numbers, l and n, are substituted into this expression).

It would be interesting to see what happens to a moving particle and how $E = \sqrt{P^2 + M^2}$ is modified. To answer this question, one must either figure out how to apply deformed Lorentz transformations to these "rest" states (and also find an analogue of boosts) or start with a different set of commuting operators: in [1] we found an alternative set of commuting operators in which $Tr_q\left(\Omega\right)$ is exchanged for $P_3 = P_{11} - P_{22}$. Such a set could probably be also used to construct representations of a massless particle[3].

Of course, one might also try to find general formulas for the states presented in this paper: we found some but could not find others. Also we did not consider here what happens when one mixes "rotations" with "oscillations" or even different kinds of "rotations" (obtained using procedures 1

---

[3]For a massless particle we must also start from a different ground state since there is no rest state in this case:

$$P_{12}\left|0\right\rangle = P_{21}\left|0\right\rangle = P_{11}\left|0\right\rangle = 0,$$

or

$$P_{12}\left|0\right\rangle = P_{21}\left|0\right\rangle = P_{22}\left|0\right\rangle = 0.$$

and 4). However it was very computationally difficult and required enormous computer resources to find the results we obtained despite the fact that the results themselves are surprisingly simple and nice. Perhaps there might exist such a point of view from which all these results are obvious and do not require such difficult calculations.

Another problem which is left open is the question of adding representations for two or more relativistic particles. The answer should be nontrivial since Γ, $\overline{\Gamma}$ and P does not appear to generate a Hopf algebra.

8 Acknowledgment

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9 Appendix: commutational relations (1) in components

9.1 $P - P$

$$
\begin{aligned}
P_{12}\ P_{11} &= P_{11}\ P_{12} + (1 - q^2)\ P_{12}\ P_{22}, \\
P_{11}\ P_{21} &= P_{21}\ P_{11} + (1 - q^2)\ P_{22}\ P_{21}, \\
P_{11}\ P_{22} &= P_{22}\ P_{11}, \\
P_{12}\ P_{21} &= P_{21}\ P_{12} + (q^2 - 1)\ P_{22}\ (P_{11} - P_{22}), \\
P_{12}\ P_{22} &= q^2 P_{22}\ P_{12}, \\
P_{22}\ P_{21} &= q^2 P_{21}\ P_{22}.
\end{aligned}
$$

9.2 $\Gamma - \Gamma$

$$
\begin{aligned}
\Gamma_{12}\Gamma_{11} &= q^2\Gamma_{11}\Gamma_{12}, \\
\Gamma_{21}\Gamma_{11} &= \frac{1}{q^2}\Gamma_{11}\Gamma_{21}, \\
\Gamma_{22}\Gamma_{11} &= \Gamma_{11}\Gamma_{22}, \\
\Gamma_{21}\Gamma_{12} &= \Gamma_{12}\Gamma_{21} + \frac{1-q^2}{q^2}\Gamma_{11}(\Gamma_{22} - \Gamma_{11}), \\
\Gamma_{22}\Gamma_{12} &= \Gamma_{12}\Gamma_{22} - \frac{1-q^2}{q^2}\Gamma_{11}\Gamma_{12}, \\
\Gamma_{22}\Gamma_{21} &= \Gamma_{21}\Gamma_{22} + \frac{1-q^2}{q^2}\Gamma_{21}\Gamma_{11}.
\end{aligned}
$$

9.3 $\overline{\Gamma} - \overline{\Gamma}$

$$\begin{aligned}
\overline{\Gamma}_{12}\overline{\Gamma}_{11} &= \overline{\Gamma}_{11}\overline{\Gamma}_{12} + (1-q^2)\overline{\Gamma}_{12}\overline{\Gamma}_{22}, \\
\overline{\Gamma}_{21}\overline{\Gamma}_{11} &= \overline{\Gamma}_{11}\overline{\Gamma}_{21} - (1-q^2)\overline{\Gamma}_{22}\overline{\Gamma}_{21}, \\
\overline{\Gamma}_{22}\overline{\Gamma}_{11} &= \overline{\Gamma}_{11}\overline{\Gamma}_{22}, \\
\overline{\Gamma}_{21}\overline{\Gamma}_{12} &= \overline{\Gamma}_{12}\overline{\Gamma}_{21} - (1-q^2)\overline{\Gamma}_{22}(\overline{\Gamma}_{22} - \overline{\Gamma}_{11}), \\
\overline{\Gamma}_{22}\overline{\Gamma}_{12} &= \frac{1}{q^2}\overline{\Gamma}_{12}\overline{\Gamma}_{22}, \\
\overline{\Gamma}_{22}\overline{\Gamma}_{21} &= q^2\overline{\Gamma}_{21}\overline{\Gamma}_{22}.
\end{aligned}$$

9.4 $\Gamma - \overline{\Gamma}$

$$\begin{aligned}
\overline{\Gamma}_{11}\Gamma_{11} &= \Gamma_{11}\overline{\Gamma}_{11} + (1-q^2)\Gamma_{12}\overline{\Gamma}_{21}, \\
\overline{\Gamma}_{12}\Gamma_{11} &= \Gamma_{11}\overline{\Gamma}_{12} + (1-q^2)\Gamma_{12}\overline{\Gamma}_{22} - (1-q^2)\overline{\Gamma}_{11}\Gamma_{12}, \\
\overline{\Gamma}_{11}\Gamma_{12} &= \Gamma_{12}\overline{\Gamma}_{11}, \\
\overline{\Gamma}_{12}\Gamma_{12} &= q^2\Gamma_{12}\overline{\Gamma}_{12}, \\
\overline{\Gamma}_{21}\Gamma_{11} &= \Gamma_{11}\overline{\Gamma}_{21}, \\
\overline{\Gamma}_{22}\Gamma_{11} &= \Gamma_{11}\overline{\Gamma}_{22} - (1-q^2)\overline{\Gamma}_{21}\Gamma_{12}, \\
\overline{\Gamma}_{21}\Gamma_{12} &= \frac{1}{q^2}\Gamma_{12}\overline{\Gamma}_{21}, \\
\overline{\Gamma}_{22}\Gamma_{12} &= \Gamma_{12}\overline{\Gamma}_{22}, \\
\overline{\Gamma}_{11}\Gamma_{21} &= \Gamma_{21}\overline{\Gamma}_{11} + (1-q^2)(\Gamma_{22} - \Gamma_{11})\overline{\Gamma}_{21}, \\
\overline{\Gamma}_{12}\Gamma_{21} &= \frac{1}{q^2}\Gamma_{21}\overline{\Gamma}_{12} + \frac{(1-q^2)}{q^2}(\Gamma_{22} - \Gamma_{11})\overline{\Gamma}_{22} - \\
&\qquad \frac{(1-q^2)}{q^2}\overline{\Gamma}_{11}(\Gamma_{22} - \Gamma_{11}), \\
\overline{\Gamma}_{11}\Gamma_{22} &= \Gamma_{22}\overline{\Gamma}_{11} - \frac{(1-q^2)}{q^2}\Gamma_{12}\overline{\Gamma}_{21}, \\
\overline{\Gamma}_{12}\Gamma_{22} &= \Gamma_{22}\overline{\Gamma}_{12} - \frac{(1-q^2)}{q^2}\Gamma_{12}\overline{\Gamma}_{22} + \frac{(1-q^2)}{q^2}\overline{\Gamma}_{11}\Gamma_{12}, \\
\overline{\Gamma}_{12}\Gamma_{22} &= \Gamma_{22}\overline{\Gamma}_{12} - \frac{(1-q^2)}{q^2}\Gamma_{12}\overline{\Gamma}_{22} + \frac{(1-q^2)}{q^2}\overline{\Gamma}_{11}\Gamma_{12}, \\
\overline{\Gamma}_{21}\Gamma_{21} &= q^2\Gamma_{21}\overline{\Gamma}_{21},
\end{aligned}$$

$$\overline{\Gamma}_{22}\Gamma_{21} = \Gamma_{21}\overline{\Gamma}_{22} - \frac{(1-q^2)}{q^2}\overline{\Gamma}_{21}(\Gamma_{22} - \Gamma_{11}),$$

$$\overline{\Gamma}_{21}\Gamma_{22} = \Gamma_{22}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{22}\Gamma_{22} = \Gamma_{22}\overline{\Gamma}_{22} + \frac{(1-q^2)}{q^2}\overline{\Gamma}_{21}\Gamma_{12}.$$

9.5 $P - \Gamma$

$$\Gamma_{11}P_{11} = qP_{11}\Gamma_{11} - (1-q^2)\Gamma_{12}P_{21},$$

$$\Gamma_{12}P_{11} = \frac{1}{q}\,P_{11}\Gamma_{12} - \frac{(1-q^2)}{q^2}\Gamma_{11}P_{12} - \frac{(1-q^2)^2}{q^2}\Gamma_{12}P_{22},$$

$$\Gamma_{11}P_{12} = \frac{1}{q}P_{12}\Gamma_{11} - (1-q^2)\Gamma_{12}P_{22},$$

$$\Gamma_{12}P_{12} = \frac{1}{q}P_{12}\Gamma_{12},$$

$$\Gamma_{21}P_{11} = qP_{11}\Gamma_{21} - (1-q^2)\Gamma_{22}P_{21} + q(1-q^2)P_{21}\Gamma_{11},$$

$$\Gamma_{22}P_{11} = \frac{1}{q}P_{11}\Gamma_{22} - \frac{(1-q^2)}{q^2}\Gamma_{21}P_{12} - \frac{(1-q^2)^2}{q^2}\Gamma_{22}P_{22} +$$
$$\frac{(1-q^2)}{q}P_{21}\Gamma_{12},$$

$$\Gamma_{21}P_{12} = qP_{12}\Gamma_{21} - (1-q^2)\Gamma_{22}P_{22} + \frac{(1-q^2)}{q}(P_{22} - P_{11})\Gamma_{11},$$

$$\Gamma_{22}P_{12} = qP_{12}\Gamma_{22} + \frac{(1-q^2)}{q}(P_{22} - P_{11})\Gamma_{12},$$

$$\Gamma_{11}P_{21} = qP_{21}\Gamma_{11},$$

$$\Gamma_{12}P_{21} = qP_{21}\Gamma_{12} - (1-q^2)\Gamma_{11}P_{22},$$

$$\Gamma_{11}P_{22} = \frac{1}{q}P_{22}\Gamma_{11},$$

$$\Gamma_{12}P_{22} = qP_{22}\Gamma_{12},$$

$$\Gamma_{21}P_{21} = \frac{1}{q}P_{21}\Gamma_{21},$$

$$\Gamma_{22}P_{21} = \frac{1}{q}P_{21}\Gamma_{22} - (1-q^2)\Gamma_{21}P_{22},$$

$$\Gamma_{21}P_{22} = \frac{1}{q}P_{22}\Gamma_{21} - \frac{(1-q^2)}{q^3}P_{21}\Gamma_{11},$$

$$\Gamma_{22}P_{22} \;=\; qP_{22}\Gamma_{22} - \frac{(1-q^2)}{q}P_{21}\Gamma_{12}.$$

9.6 $P - \overline{\Gamma}$

$$\overline{\Gamma}_{11}P_{11} \;=\; qP_{11}\overline{\Gamma}_{11} - (1-q^2)\overline{\Gamma}_{12}P_{21} + q(1-q^2)P_{12}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{12}P_{11} \;=\; \frac{1}{q}P_{11}\overline{\Gamma}_{12} + \frac{(1-q^2)}{q}P_{12}(\overline{\Gamma}_{22} - \overline{\Gamma}_{11}),$$

$$\overline{\Gamma}_{11}P_{12} \;=\; qP_{12}\overline{\Gamma}_{11} - (1-q^2)\overline{\Gamma}_{12}P_{22},$$

$$\overline{\Gamma}_{12}P_{12} \;=\; qP_{12}\overline{\Gamma}_{12},$$

$$\overline{\Gamma}_{21}P_{11} \;=\; qP_{11}\overline{\Gamma}_{21} - (1-q^2)\overline{\Gamma}_{22}P_{21},$$

$$\overline{\Gamma}_{22}P_{11} \;=\; \frac{1}{q}P_{11}\overline{\Gamma}_{22} - \frac{(1-q^2)}{q^3}P_{12}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{21}P_{12} \;=\; \frac{1}{q}P_{12}\overline{\Gamma}_{21} - (1-q^2)\overline{\Gamma}_{22}P_{22},$$

$$\overline{\Gamma}_{22}P_{12} \;=\; \frac{1}{q}P_{12}\overline{\Gamma}_{22},$$

$$\overline{\Gamma}_{11}P_{21} \;=\; \frac{1}{q}\,P_{21}\overline{\Gamma}_{11} - \frac{(1-q^2)}{q^2}\overline{\Gamma}_{21}P_{11} - \frac{(1-q^2)^2}{q^2}\overline{\Gamma}_{22}P_{21} + \frac{(1-q^2)}{q}P_{22}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{12}P_{21} \;=\; \frac{1}{q}P_{21}\overline{\Gamma}_{12} - (1-q^2)\overline{\Gamma}_{22}P_{11} + \frac{(1-q^2)}{q}P_{22}(\overline{\Gamma}_{22} - \overline{\Gamma}_{11}),$$

$$\overline{\Gamma}_{11}P_{22} \;=\; \frac{1}{q}\,P_{22}\overline{\Gamma}_{11} - \frac{(1-q^2)}{q^2}\overline{\Gamma}_{21}P_{12} - \frac{(1-q^2)^2}{q^2}\overline{\Gamma}_{22}P_{22},$$

$$\overline{\Gamma}_{12}P_{22} \;=\; qP_{22}\overline{\Gamma}_{12} - (1-q^2)\overline{\Gamma}_{22}P_{12},$$

$$\overline{\Gamma}_{21}P_{21} \;=\; qP_{21}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{22}P_{21} \;=\; qP_{21}\overline{\Gamma}_{22} - \frac{(1-q^2)}{q}P_{22}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{21}P_{22} \;=\; \frac{1}{q}P_{22}\overline{\Gamma}_{21},$$

$$\overline{\Gamma}_{22}P_{22} \;=\; qP_{22}\overline{\Gamma}_{22}.$$